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POSITIONAL l-CAPTURE IN THE GAME OF A SINGLE EVADER AND SEVERAL PURSUERS*

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A differential game of l-capture of a single evader chased by m pursuers moving with equal velocity is studied. A strategy of the pursuers is constructed depending only on the geometrical coordinates and applicable to the cases when the convex shell of initial positions of the pursuers has an empty interior.

1. Consider a differential game of m pursuers P_i and single evader E whose motions are described, respectively, by the equations

$$x_i^{\cdot} = u_i, \quad y^{\cdot} = v; \quad x_i, y, u_i, v \in R^n \tag{1.1}$$
$$|u_i| \leqslant 1, \ |v| \leqslant 1, \quad i \in I = \{1, 2, \ldots, m\}, \quad |z| = (z, z)^{1/2}$$

The pursuit is assumed terminated when the following relation is realized in the course of motion for the first time:

$$|x_i(t) - y(t)| \leqslant l \tag{1.2}$$

for at least one $i \in I$ where l is a given positive number.

Various versions of this game were discussed earlier in /1–9/ et al. Let $N_0 =$ conv $\{x_1(0), x_2(0), \ldots, x_m(0)\}$ be a convex shell of initial positions of the pursuers. It was shown in /6/ that if the set $y(0)$ belongs to the open l-neighbourhood M_0 of the set N_0, the pursuit in the game (1.1) can be terminated in a finite period of time. This also follows in a more general formulation from Theorem 2 in /8/. However, it was assumed there that at every instant $t \geqslant 0$ the pursuers knew the value $v(t)$ of the control parameter of the evader, i.e. the accessibility of the information discriminating against the evader was assumed.

Below we construct a special strategy for the pursuers based on positional information only, guaranteeing the termination of the pursuit. Moreover, the strategy U^* can also be used when the set N_0 has an empty interior (e.g. when $m \leqslant n$).

2. We will now present the basic concepts refining the formulation of the problem. Let $[a, \theta\rangle$ denote a segment $[a, \theta]$ in the case when θ is finite, and a ray $[a, +\infty)$ when $\theta = +\infty$. We consider a differential game with the following dynamic properties (the set P and Q are compact):

$$z^{\cdot} = f(z, u, v); \quad z \in R^n, \quad u \in P \subset R^p, \quad v \in Q \subset R^q \tag{2.1}$$

We shall assume that the following conditions are satisfied:

1) the function $f(z, u, v)$ is continuous in z, u, v and satisfies the Lipshitz condition in z on every compact subset R^n;

2) there is a $c \geqslant 0$, such that $(z, f(z, u, v)) \leqslant c(1 + |z|^2)$ for all $z \in R^n$, $u \in P$, $v \in Q$;

3) the terminal set $M (\subset R^n)$ is closed.

We define the strategy of the pursuer in the form of a mapping U, placing every point $z \in R^n$ in 1:1 correspondence with the pair $U(z) = (D(z), u_z(\cdot))$. Here $D(z)$ is an open region of the space R^n containing the points z, $u_z(\cdot)$: $[D(z)] \mapsto P$ is a continuously differentiable mapping (local synthesis of the pursuer controls). The symbol $[D(z)]$ denotes the closure of the set $D(z)$.

Let the initial point $z_0 = z(t_0)$ be given, a strategy U be chosen by the pursuer, and an arbitrary measurable control $v(t)$, $t \in [t_0, +\infty)$ by the evader.

We shall describe the corresponding motion of the phase point. Consider the pair $U(z_0) = (D(z_0), u_{z_0}(\cdot))$. In the first stage of the pursuit the trajectory is determined in the form of a solution of the Cauchy problem

$$z^{\cdot} = f(z, u_{z_0}(z_0), v(t)), \ z(t_0) = z_0, \ z \in [D(z_0)] \tag{2.2}$$

on the longest time interval $[t_0, \theta\rangle$ on which the inclusion $z(t) \in [D(z_0)]$ occurs. By virtue of conditions 1) and 2), the Cauchy problem has a unique solution, which can be continued to $[t_0, +\infty)$ or up to the boundary Fr $D(z_0)$ of the set $D(z_0)$. Then, two cases are possible:

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a) $\theta = +\infty$, i.e. $z(t) \in D(z_0)$ for all $t \geqslant t_0$; b) $\theta < +\infty$, i.e. $z(\theta) \in \mathrm{Fr} D(z_0)$. In case a) the motion in the first stage is completely determined. In case b) we write $z_1 = z(t_1)$ at $t = \theta$ and continue the motion of the phase point in the second stage as a solution of the Cauchy problem

$$z^{\cdot} = f(z, \quad u_{z_1}(z), \ v(t)), \ z(t_1) = z_1, \ z \in [D(z_1)]$$

where $D(z_1)$, $u_{z_1}(\cdot)$ are the components of $U(z_1)$. Repeating the above procedure we obtain the finite or infinite sequences of instants of time $t_0, t_1, \ldots, t_s, \ldots$ and subsets $D(z_0), D(z_1), \ldots, D(z_s), \ldots,$ such that the pursuer chooses his control in every interval $[t_s, t_{s+1}\rangle$ in the form of a synthesising function $u(z) = u_{z_s}(z), z \in [D(z_s)]$ and the trajectory $z(t) = z(t; z_0, U, v(\cdot))$ is determined either on the ray $[t_0, +\infty)$, or on the collection of the segments $[t_0, t_s]$.

We shall call the pair $U = (D(z), u_z(\cdot))$ (z_0, T)-admissible if it generates, under any control $v(t)$, $t \in [t_0, T\rangle$ of the evader, a trajectory $z(t; z_0, U, v(t))$ defined in the interval $[t_0, T\rangle$. We shall call the function $z(t)$ a trajectory corresponding to $z_0, U, v(\cdot)$.

The definition of the game (2.1) implies that a pursuit can be continued from the initial point z_0 up to the instant T, $T < +\infty$, provided that an (z_0, T)-admissible strategy U exists such that for any measurable control of the evader $v(\cdot)$ the trajectory $z(t)$ corresponding to z_0, U, $v(t)$, satisfies the inclusion $z(t) \in M$ at some $t \leqslant T$.

3. Let us return to the game (1.1)'. We denote by K the set of all non-empty subsets of the set I, and by $d(\pi)$ the number of elements of the set $\pi \in K$. We put $x = (x_1, x_2, \ldots, x_m)$, $z = (x_1, x_2, \ldots, x_m, y)$ noting that $z \in R^{n(m+1)}$). Further, for every $\pi \in K$ and $x \in R^n$ we assume

$$N(x, \pi) = \operatorname{conv}\{x_i, \ i \in \pi\}, \ M(x, \pi, l^*) = N(x, \pi) + l^* S$$

where S is an open unit sphere of the space R^n with centre at the origin of coordinates and l^* is a positive number. We denote by $D(x, \pi, l^*)$ the set obtained from $[M(x, \pi, l^*)]$ by deleting all sets of the form $M(x, \pi^*, l^*)$, where π^* is the characteristic subset of π, i.e.

$$D(x, \pi, l^*) = [M(x, \pi, l^*)] \diagdown \bigcup M(x, \pi^*, l^*)$$

where the union is taken over all $\pi^* \subset \pi$, $d(\pi^*) < d(\pi)$.

We assume that $|x_i(0) - y(0)| > l$ for all $\pi \in K$.

Theorem. If $y(0) \in M(x(0), \pi, l)$ for some $\pi \in K$, then the pursuit in the game (1.1) can be completed in the sense of Sect.2.

Proof. From the condition of the theorem it follows that $\pi_0 \subset \pi$ and $l_0 < l$ exist such that $y(0) \in D(x(0), \pi_0, l_0)$. We assume that $d(\pi_0) = n + 1$ (if $d(\pi_0) > n + 1$ then by the Caratheodory theorem $y(0) \in D(x(0), \pi^\circ, l_0)$ for some $\pi^\circ \subset \pi_0$, $d(\pi^\circ) = n + 1$, and the case $d(\pi_0) \leqslant n$ will be discussed below).

At the start of the motion the pursuers P_i, $i \in \pi_0$ will move with unit velocity in the direction of the initial position $y(0)$ of the evader, while the remaining pursuers will remain at their places and not take part in the game at all, i.e.

$$u_i(z) = \begin{cases} (y(0) - x_i(0))/|y(0) - x_i(0)|, & i \in \pi_0 \\ 0, & i \overline{\in} \pi_0 \end{cases} \tag{3.1}$$

Then, at some finite instant $t_1 > 0$ we shall have $y(t_1) \in \mathrm{Fr}\, D(x(t_1), \pi_0, l_0)$, by virtue of the assumption that $d(\pi_0) = n + 1$.

Consequently, for some l_1, $0 < l_1 < l$ and $\pi_0^* \subset \pi_0$, $d(\pi_0^*) < d(\pi_0)$ the relation $y(t_1) \in M(x(t_1), \pi_0^*, l_1)$ will hold, i.e. the point $y(t_1)$ will arrive at the l_1-neighbourhood of the convex shell of the smaller number of the pursuers P_i, $i \in \pi_0^*$, $d(\pi_0^*) < n + 1$. But then $\pi_1 \subset \pi_0^*$ exists such that $y(t_1) \in D(x(t_1), \pi_1, l_1)$.

Starting now from the instant $t_1 > 0$, we shall assign to the pursuers the following local synthesis defined in the region

$$D = \{z : y \in D(x, \pi_1, l_1)\}: \tag{3.2}$$

$$u_i(z) = \begin{cases} (1 - \varphi^2(h))^{1/2} e_i(z) + \varphi(z) e_0(z), & i \in \pi_1 \\ 0, & i \overline{\in} \pi_1 \end{cases}$$

$$e_0(z) = \begin{cases} h(z)/|h(z)|, & h(z) \neq 0 \\ 0, & h(z) = 0 \end{cases}$$

$$e_i(z) = \begin{cases} (x_i - x_j)/|x_i - x_j|, & i \neq j \\ 0, & i = j \end{cases}$$

$$\varphi(h) = 1 - (1 - |h|/l_1)^2$$

Here $h = h(z)$ is a vector orthogonal to $N(x, \pi_1)$ and directed towards y. The length of this vector is equal to the distance separating y from the set $N(x, \pi_1)$, j is a fixed element of π_1.

It can be shown that $(e_0(z), e_i(z)) = 0$ for all $z \in D$ and $i \in \pi_1$.

Let the evader apply any measurable control $v = v(t),\ t \geqslant t_1$. Then the equations of motion of the pursuers $P_i,\ i \in \pi_1$ and evader will take the form

$$x_i^{\cdot} = (1 - \varphi^2(h))^{1/2} e_i(z) + \varphi(h) e_0(z),\ \ x_i(t_1) = x_{i1} \tag{3.3}$$
$$y^{\cdot} = v,\ \ y(t_1) = y_1$$

Let $x_1(t), x_2(t), \ldots, x_{m_1}(t), y(t)$ be the solution of this Cauchy problem $(m_1 = d(\pi_1))$. Then the supporting plane of the convex set $N(x(t), \pi), t \geqslant t_1$ will shift, during the motion, within the space R^n, while remaining parallel to the state at $t = t_1$. Consequently, as long as $y(t) \in D(x(t), \pi_1, l_1),\ t \geqslant t_1$, the vector $e_0(z(t))$ will also remain parallel to $e_0(z(t_1))$ and the length of the vector $h(z(t))$ will be given by the formula

$$|h(z(t))| = (y(t) - x_i(t),\ e_0(t_1)),\ \ t \geqslant t_1,\ \ i \in \pi_1 \tag{3.4}$$

We find that

$$|h(z(t))| \leqslant l_1 \tag{3.5}$$

at all $t \geqslant t_1$.

Let us assume the opposite. Let the inequality (3.5) hold only in a finite time interval $[t_1, \tau_1), \tau_1 > t_1$ and $|h(z(\tau_1))| = l_1$ (such a time interval always exists, since the function $h(z)$ is continuous in z and $|h(z(t_1))| < l_1$) at the initial instant.

Then by virtue of (3.3) and (3.4) we have for all $t \in [t_1, \tau_1)$

$$d|h(z(t))|/dt = (y^{\cdot}(t),\ e_0(t_1)) - (x_i^{\cdot}(t),\ e_0(t_1)) =$$
$$(v(t),\ e_0(t_1)) - (1 - \varphi^2(h(z(t))))^{1/2}(e_i(t),\ e_0(t_1)) -$$
$$\varphi(h(z(t)))\ (e_0(t),\ e_0(t_1)) \leqslant 1 - \varphi(h(z(t))) = (1 - |h(z(t))|/l_1)^2$$

which yields

$$|h(z(t))| \leqslant l_1\{1 - [(1 - |h(z(t))|/l_1)^{-1} + t/l_1]^{-1}\} \tag{3.6}$$

From (3.6) and the continuity of the function $h(z)$ it follows that $|h(z(t))| < l$, which contradicts the assumption. Therefore the inequality (3.5) holds for all $t \geqslant t_1$.

Further, the distances between the pursuers $P_i,\ i \in \pi_1 \setminus \{j\}$ and the evader P_j tend monotonically to zero. Indeed, if we assume that $\rho_i(t) = x_j(t) - x_i(t)$, then by virtue of (3.3)

$$d\rho_i^2(t)/dt = 2(\rho_i^{\cdot}(t),\ \rho_i(t)) = -2(1 - \varphi^2(h(z(t))))^{1/2} \times |\rho_i(t)|,\ \ i \in \pi_1 \setminus \{j\}$$

Taking into account (3.6), we obtain from the last relation

$$d\rho_i^2(t)/dt = -2(1 - \varphi^2(h(z(t))))^{1/2}|\rho_i(t)| =$$
$$-2(1 - |h(z(t))|/l_1)|\rho_i(t)| \leqslant -2\{[t/l_1 +$$
$$(1 + |h(z(t_1))|/l_1)^{-1}]^{-1}\}|\rho_i(t)|$$

Integrating we obtain

$$|\rho_i(t)| \leqslant |\rho_i(t_1)| - l_1\{\ln[t/l + (1 - |h(z(t_1))|/l_1)^{-1}] - \ln[t/l + (1 - |h(z(t_1))|/l_1)^{-1}]\} \tag{3.7}$$

Inequality (3.7) implies that after some finite time $T_1 > t_1$ at least one of the functions $\rho_i(t)\ i \in \pi_1 \setminus \{j\}$ will vanish, i.e. $x_i(T_1) = x_j(T_1)$. Consequently, after a certain time $t_2 < T_1$ the evader must arrive at the boundary of the set $D(x(t_2), \pi_1, l_1)$, i.e. $y(t_2) \in FrD(x(t_2), \pi_1, l_1)$.

Therefore, at some $l_2,\ l_1 < l_2 < l$ and $\pi_2 \subset \pi_1,\ d(\pi_2) < d(\pi_1)$ the point $y(t_2)$ will arrive at the l_2-neighbourhood of the convex shell of an even smaller number of pursuers $P_i,\ i \in \pi_2$.

If $d(\pi_2) = 1$, then the pursuit is completed. If $d(\pi_2) > 1$, then from then on only the group $P_i, i \in \pi_2$ of pursuers continues the pursuit, the rest of the pursuers remaining in their places. Repeating analogous constructions for this group of pursuers we obtain, as a result, after some finite time $t_3,\ t_3 > t_2 > 0$, the inclusion

$$y(t_3) \in M(x(t_3), \pi_3, l_3),\ 0 < l_2 < l_1 < l$$
$$\pi_3 \subset \pi_2,\ \ d(\pi_3) < d(\pi_2)$$

If $d(\pi_3) = 1$, then the pursuit is completed.

When $d(\pi_3) > 1$ we continue the pursuit process described above, thus reducing the number $d(\pi_s)$ of pursuers $P_i, i \in \pi_s$ for which

$$y(t_s) \in M(x(t_s),\ \pi_s,\ l_s),\ \ 0 < l_{s-1} < l_s < l \tag{3.8}$$

where s is a positive number not greater than m.

Consequently, after the process has been repeated a finite number of times, the inclusion (3.8) will hold for a one-element set π_s, i.e., the game will be completed and this proves the theorem.

4. We shall show that the construction given in Sect.3 does not in fact define a (z_0, T)-admissible strategy with some $T < +\infty$. The equations of motiion, in the phase coordinates $z = (x_1, x_2, \ldots, x_m, y)$ of the space $R^{n(m+1)}$, take the form

$$z^{\cdot} = u + v,\ z_0 = (x_1(0),\ x_2(0),\ \ldots,\ x_m(0),\ y(0)) \tag{4.1}$$

The domains of controls for the variables u and v and the termainal set are, respectively,

$$P = \{u \in R^{n(m+1)}: u = (u_1, u_2, \ldots, u_m, 0), u_i \in R^n, |u_i| \leqslant 1\}$$
$$Q = \{v \in R^{n(m+1)}: v = (0, 0, \ldots, 0, v), v \in R^n, |v| \leqslant 1\}$$
$$M = \bigcup_{i=1}^{m} \{z \in R^{n(m+1)}: |x_i - y| \leqslant l\}$$

Let us define in the space $R^{n(m+1)}$ the set (l^* is a positive number)

$$D^*(z, \pi, l^*) = \{z \in R^{n(m+1)}: y \in D(x, \pi, l^*)\}, \pi \in K$$

If $z_0 \in D^*(z, \pi_0, l_0)$ for some $\pi_0 \in K$, $l_0 < l$, and $d(\pi_0) = n + 1$ then we can place every $z \in [D^*(z, \pi_0, l_0)]$ of the space $R^{n(m+1)}$ in 1:1 correspondence with a pair consisting of the region $D(z_0) = \{z \in R^{n(m+1)}: y \in N(x, \pi_0)\}$ and the local synthesis $u_{z_0}(z) = (u_1^\circ(z), u_2^\circ(z), \ldots, u_m^\circ(z), 0)$ where $u_i^\circ(z) = u_i(z)$ if $i \in \pi_0$ and $u_i^\circ(z) = 0$ if $i \overline{\in} \pi_0$, and the function $u_i(z)$, $i \in \pi_0$ can be found from (3.1) (first stage of the pursuit).

It follows from Sect.3 that when the pursuit takes this course, then after some finite time $t = t_1 > 0$ we shall have the inclusion $z(t_1) \in D^*(z, \pi_1, l_1)$, $0 < l_0 < l_1 < l$, $\pi_1 \subset \pi_0$, $d(\pi_1) < d(\pi_0) = n + 1$, where $z(t_1)$ at the instant $t = t_1$. Therefore if $z_0 \in D^*(z, \pi_0, l_0)$ and $d(\pi_0) < n + 1$, then the pursuit originating at the point z_0 begins in the same manner as the second stage of the pursuit in Sect.3.

We place every point $z \in [D^*(z, \pi_1, l_1)]$ ($\subset R^{n(m+1)}$) in 1:1 correspondence with a pair consisting of the region $D(z_0) = D^*(z, \pi_1, l_1)$ and local synthesis $u_{z_1}(z) = (u_1^1(z), u_2^1(z), \ldots, u_m^1(z), 0)$ where $u_i^1(z) = u_i(z)$ if $i \in \pi_1$, $u_i^1(z) = 0$ if $i \overline{\in} \pi_1$, and the functions $u_i(z)$, $i \in \pi_1$ are given by (3.2).

The discussion in Sect.3 shows that an instant of time $t = t_2$ arrives such, that $z(t_2) \in D^*(z, \pi_2, l_2)$ where $0 < l_1 < l_2 < l$, $\pi_2 \subset \pi_1$, $d(\pi_2) < d(\pi_1)$.

Furthermore, just as in the second stage, etc., we place a pair $U(z)$ in 1:1 correspondence with every point $z \in D^*(z, \pi_2, l_2)$ ($\subset R^{n(m+1)}$). Repeating the method of construction given above, we obtain finite sequences of the time instants $0, t_1, \ldots, t_s$, of the subsets $D(z_0), D(z_1), \ldots, D(z_s)$ and synthesising functions $u_{z_0}(\cdot), u_{z_1}(\cdot), \ldots, u_{z_s}(\cdot)$, which in fact determine the (z_0, T)-admissible strategy U^*.

Notes. 1°. If $z_0 \overline{\in} D^*(z, \pi_0, l_0)$ for all $\pi_0 \in K$ and $l_0, 0 < l_0 < l$ (in other words $y(0) \overline{\in} M(x(0), I, l)$), then we can show that escape is possible in the game (4.1) (or in (1.1)).

2°. If $z_0 \in \{z: y \in N(x, \pi)\}$ (or $y(0) \in N(x(0), \pi)$) for some $\pi \in K_*$ then clearly $z_0 \in D^*(z, \pi_0, \varepsilon), \pi_0 \subset \pi$ for any $\varepsilon, \varepsilon > 0$. Consequently, we have a stronger assertion for such initial positions: if $z_0 \in \{z: y \in N(x(0), \pi)\}$ (or $y(0) \in N(x(0), \pi)$) for some $\pi \in K$, then the pursuit can be completed in the game (4.1) or (1.1) in a finite time $T > 0$ in the following sense. For any positive $\varepsilon > 0$ there exists a (z_0, T)-admissible strategy such that for any measurable control of the evader $v(\cdot)$ the trajectory $z(t)$ corresponding to $z_0, U, v(\cdot)$ will satisfy the relation $|x_i(t) - y(t)| < \varepsilon$ for some value of the index $i \in \pi$ and time $t \leqslant T$.

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